The Pigeonhole Principle
Tony Liu, IMSA Math Circle
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What is this mysterious pigeonhole principle? You all know it – given 11 positive integers, two must have the same units digit; among 8 IMSA students, there are two who live in the same hall, and there are three (why?) of the same class. We can even go backwards. How many IMSA kids do we need to guarantee that there are two who live in the same hall? Eight, of course. How about to guarantee that there are three from the same class? With a bit of thinking, seven seems about right (but eight will guarantee as well). Good? Let’s get straight to some (much harder) problems. Feel free to pick and choose – work on anything that interests you. But first, here’s an example of an impossible-looking problem that shows how cool the pigeonhole principle really is.

Example 1 Suppose 51 pebbles are randomly placed inside a unit square. Show that we can always cover at least 3 pebbles with a circle of radius 1/7.

Incredible? If you’ve never seen anything like this, then it probably seems unbelievably hard at first sight. It’s actually bad at all though! Ok, so what does 51 remind us of? Start with the obvious: it’s just 1 more than 50. We have a square, so think squares. What do you know, 50 is twice 25, a perfect square! Now, all we need to do is put these observations together.

Let’s divide the unit square into 25 congruent squares with side 1/5. By pigeonhole, it shouldn’t be hard to convince yourself that one of these 25 squares has at least 3 pebbles in it. We have 51 pebbles after all, right? Now, it seems that we’ve made a lot of progress, but what does this have to do with a circle? One of the smartest guys I know always said this: if you can’t do anything smart, then do something stupid! Let’s cover the 1/2 × 1/2 square with a circle and see how big it is. Let’s see. The radius is half the diameter, which is the diagonal of the square. Is this less than 1/7? It’s up to you to finish the solution.

Hard Problems
1. Fifteen boys gathered 100 nuts. Prove that two of them gathered the same number of nuts.
2. How many people do we need to guarantee the following property: there are always three who all know each other or three who are complete strangers?
3. (IMO ’64) Each of 17 scientists corresponds with all the others. They correspond about only three topics and any two discuss exactly one topic. Prove that there are at least three scientists who correspond with each other about the same subject.
4. Each of ten segments has length between 1 and 54. Prove that we can choose three of the ten segments to form a triangle.

The idea of pigeonhole is simple. By now, you’ve probably noticed that figuring out what the pigeons and holes are is the hard part.

Even Harder Problems
5. Inside a room of area 5, you place 9 arbitrarily shaped rugs, each of area 1. Prove that there are two rugs that overlap by an area of at least 1/9.
6. Show that if we choose any n + 1 numbers from 1, 2, 3, . . . , 2n, one is divisible by another.
7. Suppose we have 10 (not necessarily distinct) integers. Prove that we can choose a subset of them with sum divisible by 10.
8. There are 650 points inside a circle of radius 16. Prove that there exists a ring with inner radius 2 and outer radius 3 covering at least ten of these points.
9. Make your own problem!

That’s it for today. Next week, we’ll probably discuss solutions to some of these problems – whichever ones you’d like to see – because they all turn out to be really neat. Feel free to ask us for hints!