Let $S$ denote the sum of three numbers on any side of the pentagon.

We first note that if we add up $S$ for every side of the pentagon, we add up the numbers on the vertices twice. However, we only add the other numbers once. In other words, in order to minimize the sum, we must place the smallest numbers on the vertices. Mathematically, we get the following inequality:

\[
5S \geq 2(1 + 2 + 3 + 4 + 5) + (6 + 7 + 8 + 9 + 10) = 70
\]

\[
S \geq 14
\]

The configuration above shows that $S = 14$ is indeed possible. ■
To simplify the algebra, let us denote
\[ a = 3x^2 + y^2 - 4y - 17 \]
\[ b = 2x^2 + 2y^2 - 4y - 6 \]
so that
\[ a - b = x^2 - y^2 - 11 \]

It follows that the original equation is equivalent to
\[
\begin{align*}
    a^3 - b^3 &= (a - b)^3 \\
    a^3 - b^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
    0 &= -3a^2b + 3ab^2 \\
    0 &= 3ab(b - a)
\end{align*}
\]

Now, we will have solutions if and only if \( a = 0, b = 0, \) or \( a = b. \)

**Case 1:** If \( a = 0, \) then
\[
\begin{align*}
    3x^2 + y^2 - 4y - 17 &= 0 \\
    3x^2 + (y - 2)^2 &= 21
\end{align*}
\]
Because the right hand side is divisible by 3, we conclude that \( 3|(y - 2)^2. \) Since squares are nonnegative, it follows that \((y - 2)^2 = 0 \) or \( 9. \) If \((y - 2)^2 = 0,\) then we have \( 3x^2 = 21, \) which has no integral solutions. If \((y - 2)^2 = 9,\) then \( y = -1 \) or \( 5, \) and \( 3x^2 = 12, \) so \( x = \pm 2. \) Our solutions \((x, y)\) are thus
\[
(-2, -1) \quad (-2, 5) \quad (2, -1) \quad (2, 5)
\]

**Case 2:** If \( b = 0, \) then
\[
\begin{align*}
    2x^2 + 2y^2 - 4y - 6 &= 0 \\
    x^2 + y^2 - 2y - 3 &= 0 \\
    x^2 + (y - 1)^2 &= 4
\end{align*}
\]
Since the only squares less than or equal to 4 are 0, 1, and \( 4, \) we must have one term equal to \( 0 \) and the other equal to \( 4. \) If \( x^2 = 0, \) then \((y - 1)^2 = 4,\) and \( y = -1 \) or \( 3. \) If \( x^2 = 4, \) then \( x = \pm 2 \) and \((y - 1)^2 = 0, \) so \( y = 1. \) Thus our solutions for this case are
\[
(0, -1) \quad (0, 3) \quad (-2, 1) \quad (2, 1)
\]

**Case 3:** If \( a = b, \) then
\[
\begin{align*}
    3x^2 + y^2 - 4y - 17 &= 2x^2 + 2y^2 - 4y - 6 \\
    x^2 - y^2 &= 11 \\
    (x + y)(x - y) &= 11
\end{align*}
\]
From this equation, we note that 11 is prime to conclude that \( x + y = \pm 1 \) or \( \pm 11, \) and \( x - y = \pm 11 \) or \( \pm 1. \) Solving these four systems of equations, we obtain the solutions
Our final solution list is thus

\[((-2, -1), (-2, 5), (2, -1), (2, 5))
,\ (0, -1)\ ,\ (0, 3)\ ,\ (-2, 1)\ ,\ (2, 1)\ ,\ (-6, -5)\ ,\ (-6, 5)\ ,\ (6, -5)\ ,\ (6, 5)\]

Since we have considered all of the cases, these are indeed all the solutions. ■
To determine to the maximum value of \( r + s + t + u \), we first make 2 claims, from which the result immediately follows.

**Claim 1:** In order for the value of \( r + s + t + u \) to be a maximum, we must have \( u = 0 \).

**Proof:** We proceed by contradiction. Assume, on the other hand, that maximum of \( r + s + t + u \) can be achieved with \( u > 0 \). We can replace \( r \) with \( r' = r + \frac{3}{4}u \) and \( u \) with \( u' = 0 \), so \( 5r' + 4s + 3t + 6u' = 5r + 4s + 3t + 6u = 100 \). This condition, along with \( r' \geq s \geq t \geq u' \geq 0 \) still holds. But now, we have \( r' + s + t + u' = r + s + t + \frac{3}{4}u > r + s + t + u \), which contradicts our original assumption.

**Claim 2:** In order for the value of \( r + s + t + u \) to be a maximum, we must have \( r = t \Rightarrow r = s = t \).

**Proof:** As before, we proceed by contradiction. If instead \( r > t \), may substitute \( r' = s' = t' = \frac{5r + 4s + 3t}{12} \), so that \( 5r' + 4s' + 3t' + 6u = 5r + 4s + 3t + 6u = 100 \). Again, the condition \( r' \geq s' \geq t' \geq u \geq 0 \) still holds. However, we now have \( r' + s' + t' + u = \frac{5r + 4s + 3t}{4} + u = r + s + t + u + \frac{r - t}{4} > r + s + t + u \), giving the desired contradiction.

From these claims, it follows that the maximum value of \( r + s + t + u \) is achieved when \( r = s = t = \frac{25}{3} \), and \( u = 0 \). Thus, the maximum value of \( r + s + t + u \) is 25.

To determine the minimum value of \( r + s + t + u \), we make a single claim which leads to the desired result.

**Claim 3:** In order for the value of \( r + s + t + u \) to be a minimum, we must have \( s = t = u = 0 \).

**Proof:** Again, we will proceed by contradiction. Assume that the the condition does not hold, so that \( \frac{1}{5}u < \frac{1}{5}s + \frac{2}{5}t \). We may substitute \( r' = r + \frac{3}{5}s + \frac{3}{5}t + \frac{6}{5}u \), and \( s' = t' = u' = 0 \). Since \( 5r' + 4s' + 3t' + 6u' = 100 \), and \( r' \geq s' \geq t' \geq u' \geq 0 \), the original conditions still hold. However, \( r' + s' + t' + u' = r + \frac{4}{5}s + \frac{3}{5}t + \frac{6}{5}u = r + s + t + u + \frac{1}{5}u - \frac{1}{5}s - \frac{2}{5}t < r + s + t + u \), proving our claim.

From this, the minimum value of \( r + s + t + u \) occurs when \( r' = r + \frac{1}{5}s + \frac{3}{5}t + \frac{6}{5}u = \frac{100}{5} = 20 \), and \( s = t = u = 0 \). Thus, the minimum value of \( r + s + t + u \) is 20. ■
Let the convex polygon have \( n \) sides. We first note that if the measures of the interior angles of the convex polygon are \( 172^\circ, 168^\circ, \ldots, [172 - 4(n - 1)]^\circ \), then the exterior angle measures are \( 8^\circ, 12^\circ, \ldots, [8 + 4(n - 1)]^\circ \). Next, we use two following well-known facts:

1. The sum of the exterior angles of any convex polygon is \( 360^\circ \).
2. The sum of the first \( n \) positive integers is \( n(n + 1)/2 \).

Thus, we have

\[
\sum_{k=1}^{n} (8 + 4(k - 1)) = 360
\]
\[
\sum_{k=1}^{n} 4(k + 1) = 360
\]
\[
n + \sum_{k=1}^{n} k = 90
\]
\[
n + \frac{n(n + 1)}{2} = 90
\]
\[
n^2 + 3n - 180 = 0
\]
\[
(n - 12)(n + 15) = 0
\]

Since \( n > 0 \), we conclude that the polygon has 12 sides. ■
Since $BE$ and $CF$ pass through $G$, we conclude that they must also be medians. It is given that $DC = BD = 1$, so $BC = 2(BD) = 2$.

Since $\triangle BGD$ is equilateral, we have $\angle BGD = 60^\circ$, and by vertical angles, $\angle AGE = \angle BGD = 60^\circ$.

Additionally, it is well-known that medians divide each other in a 2 : 1 ratio. We use this fact to obtain $AG/GD = 2$, or that $AG = 2(GD) = 2$. Similarly, $BG/GE = 2$, so $GE = \frac{1}{2}BG = \frac{1}{2}$.

Now, we apply the law of cosines to $\triangle AGE$:

\[
(AE)^2 = (AG)^2 + (GE)^2 - 2(AG)(GE) \cos \angle AGE
\]
\[
(AE)^2 = 4 + \frac{1}{4} - 2 \cos 60^\circ = \frac{13}{4}
\]
\[
AE = \frac{\sqrt{13}}{2}
\]

Since $BE$ is a median, we have $CA = 2(AE) = \sqrt{13}$.

To find $AB$, we apply the law of cosines to $\triangle AGB$.

\[
(AB)^2 = (AG)^2 + (GB)^2 - 2(AG)(GB) \cos \angle AGD
\]
\[
(AB)^2 = 4 + 1 - 4 \cos (180^\circ - \angle AGE) = 5 - 4 \cos 120^\circ = 7
\]
\[
AB = \sqrt{7}
\]

Thus, we have $AB = \sqrt{7}, BC = 2, CA = \sqrt{13}$. ■