This collection as a whole is pretty long—we’ve also included two full Olympiads. Enjoy!

Problems

1. (AMC10A ’04/18) A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

   (A) 1 (B) 4 (C) 36 (D) 49 (E) 81

2. (AIME ’94/3) The function $f$ satisfies $f(x) + f(x - 1) = x^2$ for all $x$. If $f(19) = 94$, find the remainder when $f(94)$ is divided by 1000.

3. (AIME ’02/2) Given that $P = (7, 12, 10), Q = (8, 8, 1)$ and $R = (11, 3, 9)$ are three vertices of a cube. What is its surface area?

4. (AIME ’00/4) What is the smallest positive integer with 12 positive even divisors and 6 positive odd divisors?

5. Consider a regular 8 by 8 chessboard. Clearly, we can tile the board with 1 by 2 dominoes (in a lot of ways). If we remove two opposite corners, is a tiling with 1 by 2 dominoes still possible?

6. (AIME ’95/15) A fair coin is tossed repeatedly. Find the probability of obtaining five consecutive heads before two consecutive tails.

7. (AIME ’84/15) The real numbers $x, y, z, w$ satisfy

   \[
   \frac{x^2}{2^2 - 1^2} + \frac{y^2}{2^2 - 3^2} + \frac{z^2}{2^2 - 5^2} + \frac{w^2}{2^2 - 7^2} = 1,
   \]

   \[
   \frac{x^2}{4^2 - 1^2} + \frac{y^2}{4^2 - 3^2} + \frac{z^2}{4^2 - 5^2} + \frac{w^2}{4^2 - 7^2} = 1,
   \]

   \[
   \frac{x^2}{6^2 - 1^2} + \frac{y^2}{6^2 - 3^2} + \frac{z^2}{6^2 - 5^2} + \frac{w^2}{6^2 - 7^2} = 1,
   \]

   \[
   \frac{x^2}{8^2 - 1^2} + \frac{y^2}{8^2 - 3^2} + \frac{z^2}{8^2 - 5^2} + \frac{w^2}{8^2 - 7^2} = 1.
   \]

   Find $x^2 + y^2 + z^2 + w^2$.

8. If $n + 1$ integers are selected from $1, 2, \ldots, 2n$, prove that there is one which is divisible by another.

9. (USAMO ’01) Each of eight boxes contains six balls. Each ball has been colored with one of $n$ colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer $n$ for which this is possible.
2005 United States of America Mathematical Olympiad (USAMO)

1. Determine all composite positive integers \( n \) for which it is possible to arrange all divisors of \( n \) that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

2. Prove that the system
\[
x^6 + x^3 + x^3 y + y = 147^{157} \\
x^3 + x^3 y + y^2 + y + z^9 = 157^{147}
\]
has no solutions in integers \( x, y, \) and \( z. \)

3. Let \( ABC \) be an acute-angled triangle, and let \( P \) and \( Q \) be two points on side \( BC. \) Construct point \( C_1 \) in such a way that convex quadrilateral \( APBC_1 \) is cyclic, \( QC_1 \parallel CA, \) and \( C_1 \) and \( Q \) lie on opposite sides of line \( AB. \) Construct point \( B_1 \) in such a way that convex quadrilateral \( APCB_1 \) is cyclic, \( QB_1 \parallel BA, \) and \( B_1 \) and \( Q \) lie on opposite sides of line \( AC. \) Prove that points \( B_1, C_1, P, \) and \( Q \) lie on a circle.

4. Legs \( L_1, L_2, L_3, L_4 \) of a square table each have length \( n, \) where \( n \) is a positive integer. For how many ordered 4-tuples \((k_1, k_2, k_3, k_4)\) of nonnegative integers can we cut a piece of length \( k_i \) from the end of leg \( L_i \) \((i = 1, 2, 3, 4)\) and still have a stable table? (The table is stable if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

5. Let \( n \) be an integer greater than 1. Suppose \( 2n \) points are given in the plane, no three of which are collinear. Suppose \( n \) of the given \( 2n \) points are colored blue and the other \( n \) colored red. A line in the plane is called a balancing line if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

6. For \( m \) a positive integer, let \( s(m) \) be the sum of the digits of \( m. \) For \( n \geq 2, \) let \( f(n) \) be the minimal \( k \) for which there exists a set \( S \) of \( n \) positive integers such that \( s \left( \sum_{x \in X} x \right) = k \) for any nonempty subset \( X \subset S. \) Prove that there are constants \( 0 < C_1 < C_2 \) with \( C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n. \)

2005 International Mathematical Olympiad (IMO)

1. Six points are chosen on the sides of an equilateral triangle \( ABC: A_1, A_2 \) on \( BC, B_1, B_2 \) on \( CA, \) and \( C_1, C_2 \) on \( AB, \) such that they are the vertices of a convex hexagon \( A_1A_2B_1B_2C_1C_2 \) with equal side lengths. Prove that the lines \( A_1B_2, B_1C_2, \) and \( C_1A_2 \) are concurrent.

2. Let \( a_1, a_2, \ldots \) be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer \( n \) the numbers \( a_1, a_2, \ldots, a_n \) leave \( n \) different remainders upon division by \( n. \) Prove that every integer occurs exactly once in the sequence \( a_1, a_2, \ldots. \)

3. Let \( x, y, z \) be three positive reals such that \( xyz \geq 1. \) Prove that
\[
\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.
\]

4. Determine all positive integers relatively prime to all the terms of the infinite sequence \( a_n = 2^n + 3^n + 6^n - 1, \) where \( n \) is a positive integer.
5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and $BC$ not parallel to $DA$. Let $E$ and $F$ be two variable points on the sides $BC$ and $DA$ respectively, such that $BE = DF$. The lines $AC$ and $BD$ meet at $P$, the lines $BD$ and $EF$ meet at $Q$, and the lines $EF$ and $AC$ meet at $R$. Prove that the circumcircles of the triangles $PQR$, as $E$ and $F$ vary, have a common point other than $P$.

6. In a mathematical competition in which 6 problems were posed to be participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.